

COMPLEMENTS OF RATIONALLY CONVEX SETS

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ABSTRACT. The main objective of this paper is to show that the complement of a rational convex set in \mathbb{C}^n is $(n - 2)$ -connected.

1. INTRODUCTION

A compact subset K of \mathbb{C}^n is said to be *rationally* (respectively *polynomially*) convex if for every point y_0 in the complement $\mathbb{C}^n \setminus K$ there exists a non-constant holomorphic polynomial P such that $P(y_0) = 1$ and $1 \notin P(K)$ (respectively $\|P\|_K < 1$). Notice that each polynomially convex set is rationally convex.

Rationally and polynomially convex sets play an extremely important role in complex analysis and approximation theory; we refer the reader to Stolzenberg [16] and Alexander and Wermer [2]. Nevertheless, it is usually a difficult problem to decide whether a given compact set is rationally or polynomially convex. Therefore, the results that provide topological obstructions to rational or polynomial convexity are of special interest to complex analysis. Recently, Forstnerič [8] proved via Morse theory that the complement of a polynomially convex set is simply connected. The main objective of this paper is to show that the complement of a rationally convex set is simply connected as well.

Theorem 1 (Main). *Let K be a compact rationally convex set in \mathbb{C}^n , the complement $\mathbb{C}^n \setminus K$ is simply connected for $n \geq 3$.*

We prove Theorem 1 in the second section of this paper. Notice that the complement of any compact rationally convex set K in \mathbb{C}^n is arcwise connected whenever $n \geq 2$. Actually, the complement of K has only one unbounded arcwise connected component because K is compact; and $\mathbb{C}^n \setminus K$ has no bounded arcwise connected components because it is *exhausted* by unbounded algebraic hypersurfaces $Q^{-1}(1) \subset \mathbb{C}^n$, with Q a non-constant holomorphic polynomial of several variables. Thus, we need not specify any *base point* for calculating the fundamental group of $\mathbb{C}^n \setminus K$ in Theorem 1.

Date: February 2, 2008.

1991 Mathematics Subject Classification. 32E20, 32C18 or 32Q55.

Key words and phrases. Rationally convex, Simply connected, Homotopy.

Research supported by Cinvestav and Conacyt (Mexico), and Université de Montréal (Canada).

Moreover, the proof of Theorem 1 is based in the fact that the complement $\mathbb{C}^n \setminus K$ may also be *exhausted* by smooth algebraic hypersurfaces $\mathcal{H} \subset \mathbb{C}^n$ of complex dimension $n - 1$, and so the relative pair $(\mathcal{H}, \mathcal{H} \setminus B_R)$ is arcwise and simply connected for $n \geq 3$ and B_R an open ball in \mathbb{C}^n with centre in the origin and radius $R > 0$ large enough; see Lefschetz theorem in [11, p. 552]. Suppose from now on that \mathcal{H} is an algebraic hypersurface in \mathbb{C}^n , with $n \geq 4$, and recall Theorem 5.2 of [13, p. 45]. We have that the intersection $\mathcal{H} \cap \mathcal{S}_\delta$ is $(n - 3)$ -connected for every sphere $\mathcal{S}_\delta \subset \mathbb{C}^n$ with centre in $y_0 \in \mathcal{H}$ and radius $\delta > 0$ small enough. Thus, we may consider the question whether the intersection $\mathcal{H} \cap \mathcal{S}$ is also simply connected for any arbitrary sphere $\mathcal{S} \subset \mathbb{C}^n$. A positive answer to previous question would shine light on a proof to the following possible extension of Theorem 1: The complement $\mathcal{S} \setminus K$ is simply connected for each rationally convex set K contained in a sphere $\mathcal{S} \subset \mathbb{C}^n$ with $n \geq 4$. Unfortunately, the intersection of a smooth algebraic hypersurface $\mathcal{H} \subset \mathbb{C}^n$ and a sphere $\mathcal{S} \subset \mathbb{C}^n$ is not necessarily simply connected, as the following example shows,

Example 2. Consider the polynomial $Q_2(z) = z_1^2 + z_2^2$ defined on \mathbb{C}^n with $n \geq 3$. It is easy to calculate that the hypersurface $Q_2^{-1}(1)$ is smooth, and the intersection of $Q_2^{-1}(1)$ with the unitary sphere $\{\sum |z_k|^2 = 1\}$ is equal to the one-dimensional circumference defined by the equations: $z_k = 0$ for $k \geq 3$,

$$\Re(z_1)^2 + \Re(z_2)^2 = 1 \quad \text{and} \quad \Im(z_1) = \Im(z_2) = 0.$$

In any case, it will be quite interesting to deduce whether the complement $\mathcal{S} \setminus K$ is simply connected for each rationally convex set K contained in a sphere $\mathcal{S} \subset \mathbb{C}^n$ with $n \geq 4$. On the other hand, Theorem 1 does not hold for the 2-dimensional space \mathbb{C}^2 as the following simply example shows.

Example 3. Consider the standard torus \mathcal{T}^2 in \mathbb{C}^2 defined by $|x| = |y| = 1$. It is easy to see that \mathcal{T}^2 is rationally convex. Let Υ_0 be a path parametrised by $t \mapsto (2, t) \sin(t)$ in the real plane \mathbb{R}^2 , for $0 \leq t \leq \pi$. The path Υ_0 begins at the origin, runs one time around the point $(1, 1)$ and comes back to the origin again. The compact set E composed of those points $(x, y) \in \mathbb{C}^2$ with absolute values $(|x|, |y|)$ in Υ_0 is then homeomorphic to $\Upsilon_0 \times \mathcal{T}^2$ with the fibre $\{0\} \times \mathcal{T}^2$ identified as a single point. Moreover, the intersection of \mathcal{T}^2 and E is empty, the set E is a strong deformation retract of the complement $\mathbb{C}^2 \setminus \mathcal{T}^2$, and so, the first homotopy group of $\mathbb{C}^2 \setminus \mathcal{T}^2$ is equal to the integer numbers.

We strongly recommend the works of Bredon [3] and Spanier [15] for references on homotopy theory. Besides, recalling the works of Givental' [9], Duval [6] and Duval and Sibony [7], we may construct non-orientable compact smooth submanifolds M of \mathbb{C}^2 which are rationally convex and the first homotopy groups of their complements $\mathbb{C}^2 \setminus M$ are not trivial. Finally, a direct application of algebraic topology yields the following extension of Theorem 1. We present more applications in the third section of this paper.

Corollary 4. *Let K be a compact rationally convex set in \mathbb{C}^n , its complement is $(n-2)$ -connected for $n \geq 3$:*

$$(1) \quad \pi_k(\mathbb{C}^n \setminus K) = 0 \quad \text{when} \quad 1 \leq k \leq n-2;$$

and for any commutative group G ,

$$(2) \quad H_k(\mathbb{C}^n \setminus K, G) = 0 \quad \text{when} \quad 1 \leq k \leq n-2.$$

Proof. We know that $\mathbb{C}^n \setminus K$ is arcwise and simply connected, because Theorem 1. Hence, applying Hurewicz theorem [15, p. 398], we have that equation (1) can be directly deduced from equation (2).

On the other hand, because of their definition, every rationally convex set K has a system of Stein open neighbourhoods (open rational polyhedra) $\{U_\beta\}$ in \mathbb{C}^n , and so $K = \bigcap_\beta U_\beta$. It is well known that each Stein open set $U_\beta \subset \mathbb{C}^n$ has the homotopy type of a CW -complex of real dimension less than or equal to n , see [5, p. 26] or [11, p. 548]. Whence, all cohomology groups $H^k(U_\beta, G)$ vanish for any commutative group G and $k \geq n+1$. Alexander (and Čech) cohomology groups $\bar{H}^k(K, G)$ vanish as well for $k \geq n+1$, because they are equal to the direct limits of $H^k(U_\beta, G)$ when U_β runs over a system of open neighbourhoods of K , see for example [15, p. 291] or [3, p. 348]. Finally, a direct application of Alexander duality theorem yields equation (2), as we wanted, see [15, p. 296] or [3, p. 352]. Notice that singular $H_k(\cdot)$ and reduced $\tilde{H}_k(\cdot)$ homology groups coincide for $k \geq 1$. \square

Recall that a compact rational polyhedron Π in \mathbb{C}^n is a set defined by the finite intersection,

$$(3) \quad \Pi := \overline{B} \cap \bigcap_k Q_k^{-1}(D_k),$$

where $\overline{B} \subset \mathbb{C}^n$ is any compact ball, $\{Q_k\}$ is a finite collection of holomorphic polynomials on \mathbb{C}^n , and each $D_k \subset \mathbb{C}$ is either the compact unit disk $\{|w| \leq 1\}$ or the closed ring $\{|w| \geq 1\}$. Obviously, every compact rational polyhedron is rationally convex; and by an open rational polyhedron we understand the topological interior of a compact one.

We prove Theorem 1 in the next section, and several applications are presented in the third section of this paper.

2. PROOF OF MAIN THEOREM 1

The proof is based in the following two classical results on holomorphic polynomials. Firstly, we need Lefschetz result on hyperplane sections, where it is stated that the one-point compactification of a smooth algebraic hypersurface is arcwise and simply connected. A simplified proof and several generalisations can be found in Hamm [11] or Karčauskas [12].

Theorem 5 (Lefschetz). *Let X be a non-singular (smooth) algebraic hypersurface in \mathbb{C}^n , for $n \geq 3$. There then exists a finite real number $R_0 > 0$ such that the relative pair $(X, X \setminus B_R)$ is $(n-2)$ -connected for every open ball $B_R \subset \mathbb{C}^n$ of radius $R \geq R_0$ and centre in the origin.*

This version of Lefschetz theorem can be easily deduced from the remark in [11, p. 552], we just need to observe that $X \cap B_R$ is Stein and to fix H equal to the complement of B_R . Besides, the natural existence of $R_0 > 0$ in Theorem 5 is easily understood after comparing Theorem 6.9 in [5, p. 26] with Theorem 2 or 3 in [11]. On the other hand, we also need the following result where it is shown that any non-constant holomorphic polynomial defined on \mathbb{C}^n almost induces a locally trivial fibre bundle on \mathbb{C}^n , see Verdier [17], Broughton [4] and Há Huy Vui [10].

Theorem 6. *Let $Q : \mathbb{C}^n \rightarrow \mathbb{C}$ be a non-constant holomorphic polynomial. Then, there exists a finite set Λ_Q in \mathbb{C} such that the fibres of Q induce a locally trivial fibre bundle of $\mathbb{C}^n \setminus Q^{-1}(\Lambda_Q)$ with base on $\mathbb{C} \setminus \Lambda_Q$.*

We need to combine previous two theorems into the following version of Lefschetz theorem for *fat* open sets.

Lemma 7. *Let $Q : \mathbb{C}^n \rightarrow \mathbb{C}$ be a holomorphic polynomial, for $n \geq 3$, and Ω be a bounded open set in \mathbb{C} whose compact closure $\overline{\Omega}$ is diffeomorphic to the unit disk $\{|w| \leq 1\}$ and does not intersect the finite set Λ_Q defined in Theorem 6. Given any ball B_R in \mathbb{C}^n of finite radius $R > 0$ and centre in the origin, there exists an unbounded $(n-2)$ -connected set Σ_R in \mathbb{C}^n such that $B_R \cap \Sigma_R$ is equal to the open set $Q^{-1}(\Omega) \cap B_R$.*

Proof. The result is trivial when Q is a constant polynomial, so we suppose from now on that Q is a non-constant function. The compact set $\overline{\Omega}$ is contractible in $\mathbb{C} \setminus \Lambda_Q$, for it is diffeomorphic to the unit disk, so the fibres of Q induce a global trivial fibration of $Q^{-1}(\overline{\Omega})$ with base on $\overline{\Omega}$; recall Theorem 6 and consider [1, p. 140] or [5, p. 27]. That is, given w_0 in Ω , there exists a smooth function g defined from $Q^{-1}(\overline{\Omega})$ onto the non-singular fibre $X := Q^{-1}(w_0)$, such that the pair $[Q, g]$ is a fibration diffeomorphism (it preserves fibres) from $Q^{-1}(\overline{\Omega})$ onto $\overline{\Omega} \times X$.

In particular, the restriction $g|_X$ is an auto-diffeomorphism; and we may even fix g such that $g|_X$ is the identity function. Besides, the norm $\|g(z)\|$ converges to infinity if and only if $\|z\|$ goes to infinity for z in $Q^{-1}(\overline{\Omega})$, recall that $\overline{\Omega}$ is compact. Given any finite radius $R > 0$, choose a real number $\rho \gg R$ such that the compact set $g(Q^{-1}(\overline{\Omega}) \cap \overline{B_R})$ is contained inside $X \cap B_\rho$, and so define the bounded open set,

$$(4) \quad E_\rho := [Q, g]^{-1}(\overline{\Omega} \times (X \cap B_\rho)) \subset Q^{-1}(\Omega).$$

Where $[Q, g]^{-1}$ is the inverse diffeomorphism defined from $\overline{\Omega} \times X$ onto $Q^{-1}(\overline{\Omega})$. Since $X = Q^{-1}(w_0)$ and the intersection $Q^{-1}(\Omega) \cap B_R$ is contained inside E_R , because of the way we chose $\rho \gg R$, we can deduce that the following pair of equalities holds,

$$(5) \quad \begin{aligned} E_\rho \cap B_R &= Q^{-1}(\Omega) \cap B_R, \\ X \cup E_\rho &= [Q, g]^{-1}(\{\{w_0\} \times X\} \cup [\overline{\Omega} \times (X \cap B_\rho)]). \end{aligned}$$

The open set Ω is contractible, for it is diffeomorphic to \mathbb{C} . Therefore, the spaces $X \cup E_\rho$ and X have the same homotopy type. Actually, it is easy to deduce that X is a strong deformation retract of $X \cup E_\rho$.

On the other hand, Theorem 5 implies that the pair $(X, X \setminus B_\sigma)$ is $(n-2)$ -connected for any radius $\sigma \gg \rho$ large enough. We may choose $\sigma \gg \rho$ such that $X = Q^{-1}(w_0)$ meets transversally the sphere ∂B_σ , see Theorem 6.9 in [5, p. 26]. Hence, if B_σ^\complement is the complement of B_σ in \mathbb{C}^n , we automatically have that the pair $(B_\sigma^\complement \cup X, B_\sigma^\complement)$ is $(n-2)$ -connected. Finally, we may even fix $\sigma \gg \rho$ larger enough such that the open ball B_σ contains the compact set $\overline{E_\rho}$ defined according to (4). Thus, the spaces $X \cup E_\rho$ and X have the same homotopy type and are equal outside the ball B_σ . The pair $(\Sigma_R, B_\sigma^\complement)$ is then $(n-2)$ -connected for Σ_R defined as the union $B_\sigma^\complement \cup X \cup E_\rho$. A direct application of the long exact sequence, see [1, p. 87] or [15, p. 374]

$$\rightarrow \pi_k(B_\sigma^\complement) \rightarrow \pi_k(\Sigma_R) \rightarrow \pi_k(\Sigma_R, B_\sigma^\complement) \rightarrow$$

automatically yields that Σ_R is $(n-2)$ -connected as well, recall that the complement B_σ^\complement is $(2n-2)$ -connected. It is easy to deduce that Σ_R is unbounded and arcwise connected, because all the arcwise connected components of X and $X \cup E_\rho$ are unbounded, and so they all meet the complement B_σ^\complement . Finally, since $\sigma \gg \rho \gg R$ and using equation (5), we may conclude that $B_R \cap \Sigma_R$ is equal to the open set $Q^{-1}(\Omega) \cap B_R$, as we wanted. \square

The proof of Theorem 1 follows now an inductive process. Recall the definition of a compact rationally polyhedron given in equation (3). The complement of any compact ball \overline{B} in \mathbb{C}^n is obviously simply connected. Thus, given any rationally convex set K in \mathbb{C}^n whose complement is simply connected, we only need to prove that the union of $Q^{-1}(\Omega)$ and $\mathbb{C}^n \setminus K$ is also simply connected for all open sets Ω in \mathbb{C} and holomorphic polynomials Q in \mathbb{C}^n . We prove previous statement in several steps.

Lemma 8. *Let $Q : \mathbb{C}^n \rightarrow \mathbb{C}$ be a holomorphic polynomial, for $n \geq 3$, and Π be any compact rational polyhedron in \mathbb{C}^n whose complement is simply connected. The union of $Q^{-1}(\Omega)$ and $\mathbb{C}^n \setminus \Pi$ is then simply connected for every bounded open set Ω in \mathbb{C} whose compact closure $\overline{\Omega}$ is diffeomorphic to the unit disk $\{|w| \leq 1\}$ and does not intersect the finite set Λ_Q defined in Theorem 6.*

Proof. The result is trivial whenever Q is a constant polynomial, so we suppose from now on that Q is a non-constant function. Define Π^\complement to be the complement $\mathbb{C}^n \setminus \Pi$, so the open set Π^\complement is simply connected because of the given hypotheses, and arcwise connected because Π is rationally convex. Given an arbitrary closed path (loop) Υ inside $Q^{-1}(\Omega)$ union Π^\complement , choose a ball B_R with centre in the origin and radius $R > 0$ large enough such that the compact sets Π and Υ are both contained inside B_R .

Lemma 7 implies the existence of an unbounded $(n-2)$ -connected space Σ_R in \mathbb{C}^n such that $B_R \cap \Sigma_R$ is equal to the open set $Q^{-1}(\Omega) \cap B_R$. Notice

in particular that Σ_R is both arcwise and simply connected because $n \geq 3$. Since Π is contained in B_R , the union of $Q^{-1}(\Omega)$ and $\Pi^{\mathbb{G}}$ is equal to both spaces: $\Sigma_R \cup \Pi^{\mathbb{G}}$ and the complement of $\Pi \setminus Q^{-1}(\Omega)$ in \mathbb{C}^n . We show that the loop Υ is homotopically trivial in $\Sigma_R \cup \Pi^{\mathbb{G}}$ via an alternative version of Van-Kampen's theorem [1, p. 63].

The spaces Σ_R and $\Pi^{\mathbb{G}}$ are both simply connected because of Lemma 7 and the given hypotheses, so we suppose that Υ meets Σ_R and $\Pi^{\mathbb{G}}$. Since the loop Υ is contained in B_R , we can decompose it as the finite sum $\sum_k \Upsilon_k$ of non-necessarily closed paths Υ_k contained inside either the open set $B_R \cap \Sigma_R$ or the open complement $\Pi^{\mathbb{G}}$, and moreover, such that the end points of the paths Υ_k are all in the intersection $\Sigma_R \cap \Pi^{\mathbb{G}}$. Actually, since Π is compact and Σ_R is unbounded, the space $\Sigma_R \cap \Pi^{\mathbb{G}}$ is not empty, and so we can fix a point w_0 there.

Suppose that $\Sigma_R \cap \Pi^{\mathbb{G}}$ is also arcwise connected. We can then construct closed paths (loops) T_k by joining both end points of each Υ_k to the fix point w_0 with arcs of the appropriated orientation in $\Sigma_R \cap \Pi^{\mathbb{G}}$. Every loop T_k is contained inside one of the simply connected spaces Σ_R or $\Pi^{\mathbb{G}}$. Hence, the original loop Υ is homotopic to the finite sum $\sum_k T_k$ in the union $\Sigma_R \cup \Pi^{\mathbb{G}}$, and moreover, the loops T_k are all homotopically trivial there. In other words, the union of $Q^{-1}(\Omega)$ and $\Pi^{\mathbb{G}}$ is simply connected, because it is equal to $\Sigma_R \cup \Pi^{\mathbb{G}}$ and the original loop Υ is homotopically trivial there. We just need to show now that the intersection of Σ_R and $\Pi^{\mathbb{G}}$ is indeed arcwise connected, in order to complete the proof of Lemma 8.

Recall the proof of Corollary 4: Every rationally convex set K in \mathbb{C}^n has a system of Stein open neighbourhoods, and so the Alexander cohomology group $\bar{H}^{2n-2}(K)$ vanishes for $n \geq 3$, see [5, p. 26] and [15, p. 291]. The reduced homology group $\tilde{H}_1(\mathbb{C}^n \setminus K)$ vanishes as well because Alexander duality theorem [15, p. 296]. Since $\Sigma_R \cup \Pi^{\mathbb{G}}$ is the complement in \mathbb{C}^n of the rationally convex set $\Pi \setminus Q^{-1}(\Omega)$, as it is indicated the second paragraph of this proof, we have that $\tilde{H}_1(\Sigma_R \cup \Pi^{\mathbb{G}})$ vanishes. On the other hand, the couple $\{\Sigma_R, \Pi^{\mathbb{G}}\}$ is excisive because $\Sigma_R \cup \Pi^{\mathbb{G}}$ is equal to the union of the open sets $\Pi^{\mathbb{G}}$ and $\Sigma_R \cap B_R$. Recall Lemma 7 and that B_R contains the compact set Π . Moreover, the intersection of $\Pi^{\mathbb{G}}$ and Σ_R is not empty because Σ_R is unbounded. Hence, the following exact Mayer-Vietoris sequence for reduced homology holds, see [15, p. 189] or [3, p. 229],

$$0 = \tilde{H}_1(\Sigma_R \cup \Pi^{\mathbb{G}}) \rightarrow \tilde{H}_0(\Sigma_R \cap \Pi^{\mathbb{G}}) \rightarrow \tilde{H}_0(\Sigma_R) \oplus \tilde{H}_0(\Pi^{\mathbb{G}}) \rightarrow .$$

Notice that a topological space Y is arcwise connected if and only if the reduced homology group $\tilde{H}_0(Y)$ vanishes. The fact that Π is rationally convex and Lemma 7 automatically imply that $\Pi^{\mathbb{G}}$ and Σ_R are both arcwise connected spaces, and so their intersection $\Sigma_R \cap \Pi^{\mathbb{G}}$ is also arcwise connected. We can conclude that $Q^{-1}(\Omega)$ union $\Pi^{\mathbb{G}}$ is simply connected, after recalling a paragraph above. \square

The next steep implies extending Lemma 8 to consider arbitrary open sets Ω in \mathbb{C} with a finite number of holes.

Lemma 9. *Let $Q : \mathbb{C}^n \rightarrow \mathbb{C}$ be a holomorphic polynomial, for $n \geq 3$, and Π be any compact rationally polyhedron in \mathbb{C}^n whose complement is simply connected. The union of $Q^{-1}(\Omega)$ and $\mathbb{C}^n \setminus \Pi$ is then simply connected for every open connected set Ω in \mathbb{C} with a finite number of holes.*

The fact that $\Omega \subset \mathbb{C}$ has a finite number of holes means that its fundamental group $\pi_1(\Omega)$ is free and finitely generated. Moreover, the conclusions of Lemma 9 are quite interesting when Π is a compact ball.

Proof. The result is trivial whenever Q is a constant polynomial, so we suppose from now on that Q is a non-constant function. Define Π^c to be the complement $\mathbb{C}^n \setminus \Pi$, so the open set Π^c is simply connected because of the given hypotheses, and arcwise connected because Π is rationally convex. Recall the finite set Λ_Q defined in Theorem 6. We prove this lemma by considering different open sets Ω_k in \mathbb{C} of increasing complexity.

Suppose that Ω_1 is an open set in $\mathbb{C} \setminus \Lambda_Q$ diffeomorphic to \mathbb{C} . Given any compact closed path (loop) Υ_1 inside $Q^{-1}(\Omega_1)$ union Π^c , the compact sets $Q(\Upsilon_1 \cap \Pi)$ and Λ_Q are *far away* one from each other. Therefore, there exists a bounded open set $\Omega_2 \subset \Omega_1$ such that the loop Υ_1 is contained in $Q^{-1}(\Omega_2)$ union Π^c and the compact closure $\overline{\Omega_2}$ is both diffeomorphic to the unit disk $\{|w| \leq 1\}$ and contained in Ω_1 . Notice that Λ_Q does not meet $\overline{\Omega_2}$. A direct application of Lemma 8 yields that Υ_1 is homotopically trivial in $Q^{-1}(\Omega_2)$ union Π^c and in the larger space $Q^{-1}(\Omega_1)$ union Π^c . That is, the union of $Q^{-1}(\Omega_1)$ and Π^c is simply connected.

Suppose that Ω_3 is any open connected set in $\mathbb{C} \setminus \Lambda_Q$ with a finite number of holes. We may decompose $\Omega_3 = V_1 \cup V_2$ as the union of two open sets V_k diffeomorphic to \mathbb{C} and whose intersection $V_1 \cap V_2$ has a finite number of connected components. Every open set $Q^{-1}(V_k) \cup \Pi^c$ is arcwise connected because its complement $\Pi \setminus Q^{-1}(V_k)$ is a rationally convex set. Previous paragraph implies that spaces $Q^{-1}(V_k) \cup \Pi^c$ are both simply connected, for $k = 1, 2$. Finally, notice that $\bigcap_k Q^{-1}(V_k)$ union Π^c is arcwise connected because its complement is also a rationally convex set. We just need to apply Van-Kampen's theorem, in order to deduce that $Q^{-1}(\Omega_3)$ union Π^c is simply connected as well. That is, given two arcwise and simply connected open sets, their union is simply connected whenever their intersection is arcwise connected, see [1, p. 63] or [3, p. 161].

Finally, suppose that Ω_4 is any open connected set in \mathbb{C}^n with a finite number of holes, so that Λ_Q may intersect Ω_4 . Let Υ_4 be any closed path (loop) in the open set $Q^{-1}(\Omega_4)$ union Π^c . There is a smooth loop Υ_5 homotopic to Υ_4 inside $Q^{-1}(\Omega_4)$ union Π^c such that Υ_5 meets transversally each Whitney's strata of every singular fibre $Q^{-1}(s)$, for s in $\Omega_4 \cap \Lambda_Q$, recall Corollary 1.12 in [5, p. 6] and section II-15 in [3, pp. 114-118]. The loop Υ_5

has real dimension one and every fibre $Q^{-1}(s)$ has real codimension two in \mathbb{C}^n , so the loop Υ_5 meets no singular fibre $Q^{-1}(s)$ for s in $\Omega_4 \cap \Lambda_Q$. Previous paragraph implies that Υ_5 is homotopically trivial in $Q^{-1}(\Omega_4 \setminus \Lambda_Q)$ union $\Pi^{\mathbb{C}}$, because $\Omega_4 \setminus \Lambda_Q$ has a finite number of holes. Thus, the original path Υ_4 is homotopically trivial in the larger space $Q^{-1}(\Omega_4)$ union $\Pi^{\mathbb{C}}$, and so this union is simply connected, as we wanted to show. \square

Notice that previous lemma may be extended to consider any open set Ω in \mathbb{C} . Nevertheless, the proof becomes more complicated than what we really need in this paper. We are now in position to prove Theorem 1.

Proof. (Theorem 1). We begin by proving that the complement of a rational polyhedron is simply connected. Let $\Pi \subset \mathbb{C}^n$ be a compact rational polyhedron defined according to equation (3), for $n \geq 3$. Notice that the complement of a compact ball $\mathbb{C}^n \setminus \bar{B}$ is simply connected, because it has the homotopy type of the sphere \mathcal{S}^{2n-1} . From Lemma 9, we have that the complement of the rational polyhedron $Q_1^{-1}(D_1) \cap \bar{B}$ is also simply connected, because it is equal to $Q_1^{-1}(\mathbb{C} \setminus D_1)$ union $\mathbb{C}^n \setminus \bar{B}$. Following an inductive process on the polynomials Q_k , and using Lemma 9 in every step, we can conclude that the complement $\mathbb{C}^n \setminus \Pi$ is simply connected as well.

On the other hand, let K be any compact rationally convex set in \mathbb{C}^n , and Υ a closed path in the complement $\mathbb{C}^n \setminus K$. We automatically have, from the definition, that there exists of a compact rational polyhedron $\Pi \subset \mathbb{C}^n$ which contains K and does not intersect Υ . Thus, the path Υ is homotopically trivial in $\mathbb{C}^n \setminus \Pi$, according to the previous paragraph; and so, Υ is also homotopically trivial in the larger set $\mathbb{C} \setminus K$. The complement of K is then simply connected, as we wanted. \square

3. APPLICATIONS

We want to finish this paper with the following application of Corollary 4. Recall that the rational convex hull $r(K)$ of a compact set $K \subset \mathbb{C}^n$ is equal to the intersection of all compact rational polyhedra $\Pi \subset \mathbb{C}^n$ which contains K , see for example [16] or [2]. The compact set K is obviously contained in its rationally convex hull $r(K)$. Besides, given a continuous function f defined from the q -dimensional sphere \mathcal{S}^q into an open subset $\Omega \subset \mathbb{C}^n$, we say that f is homotopically trivial in Ω if and only if it has a continuous extension to the compact $(q+1)$ -dimensional ball.

Corollary 10. *Let $K \subset \mathbb{C}^n$ be a compact rationally convex set, for $n \geq 3$, and f be a continuous function defined from the sphere \mathcal{S}^q into $\mathbb{C}^n \setminus K$, which is not homotopically trivial there. The image $f(\mathcal{S}^q)$ intersects the rationally convex hull $r(K)$ whenever $1 \leq q \leq n-2$.*

Proof. If the image $f(\mathcal{S}^q)$ does not intersect $r(K)$, then, f is homotopically trivial in the complement of $r(K)$ because Corollary 4. Hence, the function f is also homotopically trivial in the larger set $\mathbb{C}^n \setminus K$, contradiction. \square

We can deduce a similar result for homology. Recall that a q -cycle C_q in an open subset $\Omega \subset \mathbb{C}^n$ is formally a finite sum $\sum_k g_k f_k$ of continuous functions f_k defined from the standard q -dimensional simplex Δ^q into Ω , and coefficients g_k in a commutative group G , see [15] or [3]. We define the image of C_q as the union of all $f_k(\Delta^q)$ for the coefficients $g_k \neq 0$.

Corollary 11. *Let $K \subset \mathbb{C}^n$ be a compact rationally convex set, for $n \geq 3$, and C_q be a q -dimensional cycle in the complement $\mathbb{C}^n \setminus K$, which is not homologous to zero there. The image of C_q intersects the rationally convex hull $r(K)$ whenever $1 \leq q \leq n - 2$.*

There exists a basic difference between homotopy and homology. Given a continuous function f defined from \mathcal{S}^q into an open set $\Omega \subset \mathbb{C}^n$. We can express \mathcal{S}^q as the union of two compact hemispheres D and E , and so we have that $C_q := f|_D + f|_E$ is a q -cycle. The cycle C_q is homologous to zero in Ω , whenever f is homotopically trivial there. Nevertheless, the cycle C_q may be homologous to zero, even when f is not homotopically trivial.

Moreover, the difference between the first homotopy group $H_1(\cdot)$ and the fundamental group $\pi_1(\cdot)$ is essential, even when $H_1(\cdot)$ is the abelianisation of $\pi_1(\cdot)$. For example, recalling that an arc Υ is homeomorphic to the compact unit interval $[0, 1]$ in the real line, we automatically have that the Alexander cohomology groups $\bar{H}^k(\Upsilon)$ vanish for every $k \geq 1$, and so the first homology group $H_1(\mathbb{C}^n \setminus \Upsilon)$ vanishes for $n \geq 1$ as well because Alexander's duality theorem [3, p. 352]. Nevertheless, Rushing [14, §2.6] produces several examples of arcs Υ in \mathbb{C}^n whose complement is not simply connected. In particular, Rushing work produces examples of arcs (and copies of the Cantor set) which cannot be rationally convex sets, according to Theorem 1.

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